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Normal frames for derivations and linear connections and the equivalence principle

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Abstract

Frames normal for derivations of the tensor algebra over a manifold and for linear connections in vector bundles are defined and studied. In particular, such frames exist at every fixed point and/or along injective path. Inertial frames for gauge fields are introduced and on this ground the principle of equivalence for (system of) gauge fields is formulated.

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1. Introduction

Until 1992, the existence of normal frames and coordinates was known at a single point and along injective paths only for symmetric linear connections on a manifold; besides, a necessary and sufficient condition for the existence of these frames on submanifolds for the mentioned connections was found [1]. The papers [2–4] (see also their early versions [5–7]) completely solved the problems of existence, uniqueness and holonomicity of frames normal on submanifolds for derivations of the tensor algebra over a manifold, in particular

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for arbitrary, with or without torsion, linear connections on a manifold. At last, these results were generalized in [8] for linear transports along paths in vector bundles. The present work can be considered as a review, continuation and application of the cited references.

Section 2 of the work, consisting of Sections 2.1–2.5, is devoted to frames normal for derivations of the tensor algebra over a manifold. It is an updated review of [2–4]. In Section 3, consisting of Sections 3.1–3.5, are investigated similar problems concerning arbitrary linear connections in vector bundles.

In Section 2.1 are reviewed some basic concepts regarding derivations of the tensor algebra over a differentiable manifold and the general concept of a frame normal for them is introduced. Section 2.2 is devoted to the general existence, uniqueness and holonomicity of frames normal for derivations along (arbitrary) vector fields. A necessary and sufficient condition for existing of such frames along (locally) injective mappings is proved. The results obtained are specialized at a single point, along (locally) injective paths and on neighborhoods. Similar problems concerning frames normal for derivations along a fixed vector field are studied in Section 2.3. Special attention on frames normal for derivations along smooth paths is paid in Section 2.4. Some basic results of Sections 2.1–2.4 are specialized for linear connections on a manifold in Section 2.5.

Section 3.1 recalls the most suitable definition of a linear connection in a vector bundle and some consequences of it. Section 3.2 summarizes basic concepts of the theory of linear transports along paths in vector bundles. In Section 3.3 are proved necessary and sufficient conditions for a derivation or a linear transport along paths in vector bundles to define a linear connection. An explicit bijective correspondence between a particular class of such objects and the set of linear connections is derived. The parallel transports generated by linear connections are described in terms of linear transports along paths.

In Section 3.4, the frames normal for linear connections in vector bundles are defined and the basic equation responsible for their existence and properties is derived. Since this equation coincides with similar equation investigated in Section 2, the conclusion is made that the results of Section 2 can mutatis mutandis be applied to solve similar problems concerning frames normal for linear connections in vector bundles. Some particular results are written explicitly.

In Section 3.5 is shown how inertial frames in gauge field theories should be introduced. The principle of equivalence, which in fact is a theorem, for a particular gauge field is formulated. An example is presented for the introduction of inertial frames and formulation of the equivalence principle for a system of gauge fields (and, possibly, gravitational one). Section 4 gives the conclusion.

2. Normal frames for derivations of the tensor algebra over a manifold

The present, second, section of this investigation is a revised, updated and unified version of the material in [2–4], where references to original papers are given. The introductory text, concerning derivations of the tensor algebra over a manifold, is abstracted mainly from [9], where further details can be found. Frames normal for such derivations are defined and studied, at first, in the general case on arbitrary subsets/submanifolds and, then, the results obtained are specialized for more particular cases.

2.1. Derivations, their components, curvature and torsion

A derivation of the tensor algebra $\mathcal{T}(M)$ over a differentiable manifold M, $\dim M < \infty$, is a linear mapping $D : \mathcal{T}(M) \to \mathcal{T}(M)$ which satisfies the Leibnitz differentiation rule with respect to the tensor product, preserves the tensor's type, and commutes with the contractions of tensor fields [10, Ch. I, Section 3]. By [10, Proposition 3.3 of Chapter I] any D admits a unique representation in the form $D = L_X + S$ for some (unique for a given D) vector field X and tensor field S of type $(1, 1)$. Here L_X is the Lie derivative along X [9] and S is considered as a derivation of $\mathcal{T}(M)$ [10], which for a covariant derivative ∇ is given through $S_X(Y) = \nabla_X(Y) - [X, Y]$, Y being a vector field and $[X, Y] := X \circ Y - Y \circ X$. Since the dependence of D on X will be important further, we shall write D_X for D, with $D = L_X + S$, and say that D_X is a *derivation along* X; a mapping $D : X \mapsto L_X + S$ will be called a *derivation along vector fields*, an example being $\nabla : X \mapsto \nabla_X$.

Let $\{E_i, i = 1, \dots, n := \dim(M)\}$ be a (coordinate or not [11]) local frame (field of bases) of vector fields in the tangent to M bundle. It is holonomic (anholonomic) if the vectors $E_1, \dots, E_n$ commute (do not commute) [11]. Let T be a C^1 tensor field of type (p, q), p and q being integers or zero(s), with local components $T^{i_1,\dots,i_p}_{j_1,\dots,j_q}$ with respect to the tensor frame associated with $\{E_i\}$. Here and below all Latin indices, maybe with some super- or subscripts, run from 1 to $n := \dim(M)$. Using the explicit action of L_X and S_X on tensor fields [10] and the usual summation rule about indices repeated on different levels, we find the components of $D_X T$ to be

$$(D_X T)^{i_1,\dots,i_p}_{j_1,\dots,j_q} = X(T^{i_1,\dots,i_p}_{j_1,\dots,j_q}) + \sum_{a=1}^{p} (\Gamma_X)^{i_a}_{k} T^{i_1,\dots,i_{a-1} k i_{a+1},\dots,i_p}_{j_1,\dots,j_q} - \sum_{b=1}^{q} (\Gamma_X)^{k}_{j_b} T^{i_1,\dots,i_p}_{j_1,\dots,j_{b-1} k j_{b+1},\dots,j_q}. \quad (2.1)$$

Here $X(f)$ denotes the action of $X = X^i E_i$ on a C^1 scalar function f, i.e. $X(f) = X^k E_k(f)$ and the explicit form of Γ_X is

$$(\Gamma_X)^i_j = (S_X)^i_j - E_j(X^i) + C^i_{kj} X^k, \quad (2.2)$$

where C^i_{kj} define the commutators of the basic vector fields by $[E_j, E_k] = C^i_{jk} E_i$. We call $(\Gamma_X)^i_j$ the *components* of D_X. In particular, we have

$$D_X(E_j) = (\Gamma_X)^i_j E_i. \quad (2.3)$$

The change $\{E_i\} \mapsto \{E'_m := A^i_m E_i\}$, $A := [A^i_m]$ being a non-degenerate matrix function, implies the transformation of $(\Gamma_X)^i_j$ into (see (2.3)) $(\Gamma'_X)^m_l = (A^{-1})^m_i A^j_l\ (\Gamma_X)^i_j + (A^{-1})^m_i X(A^i_l)$. Introducing the matrices $\Gamma_X := [(\Gamma_X)^i_j]$ and $\Gamma'_X := [(\Gamma'_X)^m_l]$ and putting $X(A) := X^k E_k(A) = [X^k E_k(A^i_m)]$, we get

$$\Gamma'_X = A^{-1}\{\Gamma_X A + X(A)\}. \quad (2.4)$$

If ∇ is a linear connection with local coefficients Γ^i_{jk} (see, e.g. [10,12,13]), then $\nabla_X(E_j) = (\Gamma^i_{jk}X^k)E_i$ [10]. Hence, we see from (2.3) that D_X is a covariant derivative along X iff

$$(\Gamma_X)^i_j = \Gamma^i_{jk}X^k \tag{2.5}$$

for some functions Γ^i_{jk}.

Let D be a derivation along vector fields and X and Y be vector fields. The *torsion operator* T^D of D and the *curvature operator* R^D of a C^1 derivation D (i.e. $(\Gamma_X)^i_j$ are C^1 functions) are defined respectively by

$$\begin{aligned} T^D(X,Y) &:= D_XY - D_YX - [X,Y],\\ R^D(X,Y) &:= D_X \circ D_Y - D_Y \circ D_X - D_{[X,Y]}. \end{aligned} \tag{2.6}$$

A derivation D is *torsion-free* (resp. *curvature-free* or *flat*) on a set $U \subseteq M$ if $T^D = 0$ (resp. $R^D = 0$) on this set (cf. [10]).

Using the equation $D_X = L_X + S_X$, one finds the following representations for the curvature and torsion operators:

$$\begin{aligned} R^D(X,Y) &= S_X \circ S_Y - S_Y \circ S_X + [X, S_Y\cdot] - [Y, S_X\cdot]\\ &\qquad + S_X([Y,\cdot]) - S_Y([X,\cdot]) - S_{[X,Y]},\\ T^D(X,Y) &= S_X(Y) - S_Y(X) + [X,Y]. \end{aligned}$$

We have for them, respectively, the following local expressions:

$$[(R^D(X,Y))^i_j] = X(\Gamma_Y) - Y(\Gamma_X) + \Gamma_X\Gamma_Y - \Gamma_Y\Gamma_X - \Gamma_{[X,Y]}, \tag{2.7}$$

$$(T^D(X,Y))^i = (\Gamma_X)^i_jY^j - (\Gamma_Y)^i_jX^j - C^i_{jk}X^jY^k. \tag{2.8}$$

For a linear connection ∇ is fulfilled $(R^\nabla(X,Y))^i_j = R^i_{jkl}X^kY^l$ and $(T^\nabla(X,Y))^i = T^i_{kl}X^kY^l$, where R^i_{jkl} and T^i_{kl} are the components of the usual curvature and torsion tensors, respectively [10,11].

Further in this work we shall investigate a special class of frames (fields of bases) which are singled out by the following definition.

Definition 2.1. Given a derivation D_X of the tensor algebra over a manifold M and a subset $U \subseteq M$. A *frame* $\{E_i\}$ defined over an open subset V of M containing U or equal to it, $V \supseteq U$, is called *normal for D_X over U* if in it the components of D_X vanish everywhere on U. A frame is *normal* for a derivation $D : X \mapsto D_X$ along vector fields, if it is normal for D_X for every X. Respectively, $\{E_i\}$ is *normal for D_X along a mapping* $g : Q \to M$, $Q \neq \varnothing$, if $\{E_i\}$ is normal for D_X over $g(Q)$.

All of the information about normal frames, e.g. their existence and uniqueness, is encoded in the transformation equation (2.4). Indeed, given an arbitrary frame $\{E_i\}$ over $V \supseteq U$, a frame $\{E'_i = A^j_iE_j\}$ is normal for D_X over U iff $\Gamma_{X'^i_j}|_U = 0$, which, by virtue of (2.4), is equivalent to

$$(\Gamma_XA + X(X))|_U = 0. \tag{2.9}$$

The properties of the solutions of this equation relative to $A = [A_i^j]$, if any, are responsible for all properties of the frames normal for D_X on U. We call this (matrix) equation the *equation of the frames normal for D_X on U* or simply the *normal frame(s) equation* (for D_X on U). The exploration of this equation is the main purpose of what follows below in the present part of our investigation.

2.2. Normal frames for derivations along vector field

Let U be a subset of a manifold M and D be a derivation along vector fields, i.e. D_X to be a derivation along *arbitrary (every)* vector field X on M. Problems concerning existence, uniqueness and holonomicity of frames normal for D on U will be studied bellow.

Definition 2.2. A derivation D (resp. D_X) along vector fields (resp. along a vector field X) is called *linear* on $U \subseteq M$ if in some (and hence in any—see (2.4)) frame $\{E_i\}$, defined on U or on a larger set, is fulfilled

$$\Gamma_X(x) = \Gamma_k(x) X^k(x) \tag{2.10}$$

for every (resp. the given) vector field X. Here $x \in U$, $X = X^k E_k$, and Γ_k are some matrix-valued functions on U.

Evidently, a linear connection (covariant derivative) $\nabla : X \mapsto \nabla_X$ is a derivation linear on any $U \subseteq M$ (see (2.5)). The importance of Definition 2.2 in the theory of normal frames is established by the following result.

Proposition 2.1. *If for some derivation D along vector fields (resp. D_X along a vector field X), there exists a frame normal for it on $U \subseteq M$, then D (resp. D_X) is linear on U, i.e., the linearity of a derivation on a set is a necessary condition for the existence of frame(s) normal for it on U.*

Proof. Let us fix a frame $\{E_i\}$ and put $E'_i = A_i^j E_j$. Then $\Gamma'_X|_U = 0$, i.e. $\Gamma'_X(x) = 0$ for $x \in U$, which, in conformity with (2.4), is equivalent to (2.10) with $\Gamma_k = -(E_k(A))A^{-1}$, $A = [A_j^i]$. □

The opposite statement to Proposition 2.1 is generally not true and for its appropriate formulation we need some preliminary results and explanations.

Let p be an integer, $p \geq 1$, and the Greek indices α and β run from 1 to p. Let J^p be a neighborhood in $\mathbb{R}^p$ and $\{s^\alpha\} = \{s^1, \ldots, s^p\}$ be Cartesian coordinates in $\mathbb{R}^p$.

Lemma 2.1. *Let $Z_\alpha : J^p \to \mathrm{GL}(m, \mathbb{R})$, $\mathrm{GL}(m, \mathbb{R})$ being the group of $m \times m$ matrices on $\mathbb{R}$, be C^1 matrix-valued functions on J^p. Then the initial-value problem*

$$\left.\frac{\partial Y}{\partial s^\alpha}\right|_s = Z_\alpha(s) Y, \quad Y|_{s=s_0} = \mathbb{1}, \quad \alpha = 1, \ldots, p, \tag{2.11}$$

where $\mathbb{1} := [\delta_j^i]_{i,j=1}^m$ is the identity (unit) matrix of the corresponding size, $s \in J^p$, $s_0 \in J^p$ is fixed, and Y is $m \times m$ matrix function on J^p, has a solution, denoted by $Y =$

$Y(s, s_0; Z_1, \dots, Z_p)$, which is unique and smoothly depends on all its arguments if and only if

$$R_{\alpha\beta}(Z_1, \dots, Z_p) := \frac{\partial Z_\alpha}{\partial s^\beta} - \frac{\partial Z_\beta}{\partial s^\alpha} + Z_\alpha Z_\beta - Z_\beta Z_\alpha = 0. \tag{2.12}$$

Proof. According to the results from [14, Chapter VI], in which $Z_1, \dots, Z_p$ are of class C^1, the integrability conditions for (2.11) are (cf. [14, Chapter VI, Eq. (1.4)])

$$\begin{aligned} 0 &= \frac{\partial^2 Y}{\partial s^\alpha \partial s^\beta} - \frac{\partial^2 Y}{\partial s^\beta \partial s^\alpha} = \frac{\partial (Z_\beta Y)}{\partial s^\alpha} - \frac{\partial (Z_\alpha Y)}{\partial s^\beta} \\ &= \frac{\partial Z_\beta}{\partial s^\alpha} Y - \frac{\partial Z_\alpha}{\partial s^\beta} Y + Z_\beta Z_\alpha Y - Z_\alpha Z_\beta Y = -R_{\alpha\beta}(Z_1, \dots, Z_p) Y. \end{aligned}$$

Hence (see, e.g. [14, Chapter VI, Theorem 6.1]) the initial-value problem (2.11) has a unique solution (of class C^2) iff (2.12) is satisfied. □

Let $p \le n := \dim(M)$, $\alpha, \beta = 1, \dots, p$ and $\mu, \nu = p+1, \dots, n$. Let $\gamma : J^p \to M$ be a C^1 mapping. We suppose that for any $s \in J^p$ there exists its (p-dimensional) neighborhood $J_s \subseteq J^p$ such that the restricted mapping $\gamma|_{J_s} : J_s \to M$ is without self-intersections (is injective), i.e. in J_s do not exist points s_1 and $s_2 \neq s_1$ with the property $\gamma(s_1) = \gamma(s_2)$. This assumption is equivalent to the one that the points of self-intersections of γ, if any, can be separated by neighborhoods. For brevity, we call such mappings *locally injective*. With J_s^p, we denote the union of all the neighborhoods J_s with the above property; evidently, J_s^p is the maximal neighborhood of s in which γ is injective.

Let us suppose at first that $J_s^p = J^p$, i.e. that γ is without self-intersection, and that $\gamma(J^p)$ is contained in a single coordinate neighborhood V of M.

Let us fix some one-to-one onto C^1 mapping $\eta : J^p \times J^{n-p} \to M$ such that $\eta(\cdot, \mathbf{t}_0) = \gamma$ for a fixed $\mathbf{t}_0 \in J^{n-p}$, i.e. $\eta(s, \mathbf{t}_0) = \gamma(s)$, $s \in J^p$. In $V \cap \eta(J^p, J^{n-p})$, we define coordinates $\{x^i\}$ by putting $(x^1(\eta(s, \mathbf{t})), \dots, x^n(\eta(s, \mathbf{t}))) := (s, \mathbf{t}) \in \mathbb{R}^n$, $s \in J^p$, $\mathbf{t} \in J^{n-p}$.

Proposition 2.2. *Let $\gamma : J^p \to M$ be a C^1 injective mapping such that $\gamma(J^p)$ lies only in one coordinate neighborhood. Let a derivation D along vector fields be linear on $\gamma(J^p)$. Then a necessary and sufficient condition for the existence of a frame $\{E'_i\}$, defined in a neighborhood of $\gamma(J^p)$, which is normal for D on $\gamma(J^p)$ is the validity in the above-defined coordinates $\{x^i\}$ of the equalities*

$$[R_{\alpha\beta}(-\Gamma_1 \circ \gamma, \dots, -\Gamma_p \circ \gamma)]|_{J^p} = 0, \qquad \alpha, \beta = 1, \dots, p, \tag{2.13}$$

where $R_{\alpha\beta}(\cdots)$ are defined by (2.12) for $m = n$ and $(s^1, \dots, s^p) = s \in J^p$, i.e.

$$\begin{aligned} &[R_{\alpha\beta}(\Gamma_1 \circ \gamma, \dots, \Gamma_p \circ \gamma)](s) \\ &\quad = \frac{\partial \Gamma_\alpha(\gamma(s))}{\partial s^\beta} - \frac{\partial \Gamma_\beta(\gamma(s))}{\partial s^\alpha} + (\Gamma_\alpha \Gamma_\beta - \Gamma_\beta \Gamma_\alpha)|_{\gamma(s)}. \end{aligned} \tag{2.14}$$

Remark 2.1. This result was obtained by means of another method in [1] for the special case when D is a symmetric affine connection and U is a submanifold of M.

Proof. The following considerations will be done in the above-defined coordinate neighborhood $V \cap \eta(J^p, J^{n-p})$ and coordinates $\{x^i\}$. Let $E_i = \partial/\partial x^i$.

NECESSITY. Let there exists a normal frame $\{E'_i = A_i^j E_i\}$ on $\gamma(J^p)$, i.e. $\Gamma'_X(\gamma(s)) = 0$, $s \in J^p$. By (2.4), the existence of $\{E'_i\}$ is equivalent to that of $A = [A_i^j]$, transforming $\{E_i\}$ into $\{E'_i\}$, and such that $[A^{-1}(\Gamma_X A + X(A))]|_{\gamma(s)} = 0$ for *every* X. As D is linear on $\gamma(J^p)$ (cf. Proposition 2.1), Eq. (2.10) is valid for $x \in \gamma(J^p)$ and some matrix-valued functions Γ_k. Consequently A must be a solution of $\Gamma'_k(x) = 0$, i.e. of

$$\Gamma_k(\gamma(s))A(\gamma(s)) + \left.\frac{\partial A}{\partial x^k}\right|_{\gamma(s)} = 0, \qquad s \in J^p. \tag{2.15}$$

Now define non-degenerate matrix-valued functions B and B_i by

$$A(\gamma(s)) = B(s), \qquad \left.\frac{\partial A}{\partial x^\alpha}\right|_{\gamma(s)} = \frac{\partial B(s)}{\partial s^\alpha}, \quad \alpha = 1, \dots, p, \qquad \left.\frac{\partial A}{\partial x^\nu}\right|_{\gamma(s)} = B_\nu(s),$$
$$\nu = p+1, \dots, n.$$

Substituting these equalities into (2.15), we see that it splits into

$$\Gamma_\alpha(\gamma(s))B(s) + \frac{\partial B(s)}{\partial s^\alpha} = 0, \quad \alpha = 1, \dots, p, \tag{2.16}$$

$$\Gamma_\nu(\gamma(s))B(s) + B_\nu(s) = 0, \quad \nu = p+1, \dots, n. \tag{2.17}$$

As these equations do not involve B_α, the B_α's are left arbitrary by (2.15), while the remaining B_i's are expressed via $B(s)$ through (see (2.17))

$$B_\nu(s) = -\Gamma_\nu(\gamma(s))B(s), \quad \nu = p+1, \dots, n. \tag{2.18}$$

So, $B(s)$ is the only quantity for determination. It must satisfy (2.16). If we arbitrary fix the value $B(s_0) = B_0$ for a fixed $s_0 \in J^p$ and put $Y(s) = B(s)B_0^{-1}$ (B is a non-degenerate as A is such by definition), we see that Y is a solution of the initial-value problem

$$\left.\frac{\partial Y}{\partial s^\alpha}\right|_s = -\Gamma_\alpha(\gamma(s))Y(s), \quad \alpha = 1, \dots, p, \quad Y|_{s=s_0} = \mathbb{1}_p = [\delta^i_j]^p_{i,j=1}. \tag{2.19}$$

By Lemma 2.1, this initial-value problem has a unique solution, given by $Y = Y(s, s_0; -\Gamma_1 \circ \gamma, \dots, -\Gamma_p \circ \gamma)$, iff the integrability conditions (2.13) are valid.

Consequently the existence of $\{E'_i\}$ (or of A) leads to (2.13).

SUFFICIENCY. If (2.13) takes place, the general solution of (2.16) is

$$B(s) = Y(s, s_0; -\Gamma_1 \circ \gamma, \dots, -\Gamma_p \circ \gamma)B_0, \tag{2.20}$$

in which $s_0 \in J^p$ and the non-degenerate matrix B_0 are fixed. Consequently, admitting A to be a C^1 matrix-valued function, we see that in $V \cap \eta(J^p, J^{n-p})$ the matrix function $A(\eta(s, \mathbf{t}))$, $s \in J^p$, $\mathbf{t} \in J^{n-p}$ can be expanded up to second order terms with respect to $(\mathbf{t} - \mathbf{t}_0)$ as

$$\begin{aligned} A(\eta(s,\mathbf{t})) = {} & B(s) + B_i(s)[x^i(\eta(s,\mathbf{t})) - x^i(\eta(s,\mathbf{t}_0))] + B_{ij}(s,\mathbf{t};\eta)[x^i(\eta(s,\mathbf{t})) \\ & - x^i(\eta(s,\mathbf{t}_0))][x^j(\eta(s,\mathbf{t})) - x^j(\eta(s,\mathbf{t}_0))] \end{aligned} \tag{2.21}$$

for the above-defined matrix-valued functions B, B_i, and some B_{ij}, which are such that $\det B(s) \neq 0, \infty$ and B_{ij} and their first derivatives are bounded when $\mathbf{t} \to \mathbf{t}_0$. (Note that in (2.21) the terms corresponding to $i, j = 1, \ldots, p$ are equal to zero due to the definition of $\{x^i\}$.) In this case, due to (2.16)–(2.20), the general solution of (2.15) is

$$\begin{aligned} A(\eta(s,\mathbf{t})) = {} & \left\{ \mathbb{1} - \sum_{\lambda=p+1}^{n} \Gamma_\lambda(\gamma(s))[x^\lambda(\eta(s,\mathbf{t})) - x^\lambda(\gamma(s))] \right\} \\ & \times Y(s, s_0; -\Gamma_1 \circ \gamma, \ldots, -\Gamma_p \circ \gamma) B_0 \\ & + \sum_{\mu,\nu=p+1}^{n} \{ B_{\mu\nu}(s,\mathbf{t};\eta) \\ & \times [x^\mu(\eta(s,\mathbf{t})) - x^\mu(\gamma(s))][x^\nu(\eta(s,\mathbf{t})) - x^\nu(\gamma(s))]\}, \end{aligned} \tag{2.22}$$

where $s_0 \in J^p$ and the non-degenerate matrix B_0 are fixed and $B_{\mu\nu}$, $\mu, \nu = p+1, \ldots, n$, together with their first derivatives are bounded when $\mathbf{t} \to \mathbf{t}_0$. (The fact that into (2.22) enter only sums from $p+1$ to n is a consequence from $x^\alpha(\eta(s,\mathbf{t})) = x^\alpha(\gamma(s)) = s^\alpha$, i.e. $x^\alpha(\eta(s,\mathbf{t})) - x^\alpha(\eta(s,\mathbf{t}_0)) = x^\alpha(\eta(s,\mathbf{t})) - x^\alpha(\gamma(s)) = s^\alpha - s^\alpha \equiv 0$, $\alpha = 1, \ldots, p$.)

Hence, from (2.13) follows the existence of a class of matrices $A(x)$, $x \in V \cap \eta(J^p, J^{n-p})$ such that the frames $\{E'_i = A^j_i E_j\}$ are normal for D (which is supposed to be linear on $\gamma(J^p)$). □

Thus frames $\{E'_i\}$ in which $\Gamma'_X = 0$ exist iff (2.13) is satisfied. If (2.13) is valid, then the normal frames $\{E'_i\}$ are obtained from $\{E_i = \partial/\partial x^i\}$ by means of linear transformations whose matrices must have the form (2.22).

Now we are ready to consider a general smooth (C^1) locally injective mapping $\gamma : J^p \to M$, i.e. such that its points of self-intersection, if any, can be separated by neighborhoods. For any $r \in J^p$ choose a coordinate neighborhood $V_{\gamma(r)}$ of $\gamma(r)$ in M. Let there be given a fixed C^1 bijective mapping $\eta_r : J^p_r \times J^{n-p} \to M$ such that $\eta_r(\cdot, \mathbf{t}^r_0) = \gamma|_{J^p_r}$ for some $\mathbf{t}^r_0 \in J^{n-p}$. In the neighborhood $V_{\gamma(r)} \cap \eta_r(J^p_r, J^{n-p})$ of $\gamma(J^p_r) \cap V_{\gamma(r)}$ we introduce local coordinates $\{x^i_r\}$ defined by

$$(x^1_r(\eta_r(s,\mathbf{t})), \ldots, x^n_r(\eta_r(s,\mathbf{t}))) := (s,\mathbf{t}) \in \mathbb{R}^n,$$

where $s \in J^p_r$ and $\mathbf{t} \in J^{n-p}$ are such that $\eta_r(s,\mathbf{t}) \in V_{\gamma(r)}$.

Theorem 2.1. *Let the points of self-intersection of a C^1 mapping $\gamma : J^p \to M$, if any, be separable by neighborhoods. Let a derivation D along vector fields be linear on $\gamma(J^p)$, i.e. (2.10) to be valid for any X and $x \in \gamma(J^p)$. Then a necessary and sufficient condition for the existence in some neighborhood of $\gamma(J^p)$ of a frame $\{E'_i\}$ normal for D (along every vector field) on $\gamma(J^p)$ is for every $r \in J$ in the above-defined local coordinates $\{x^i_r\}$ to be fulfilled*

$$[R_{\alpha\beta}(-\Gamma_1 \circ \gamma, \ldots, -\Gamma_p \circ \gamma)](s) = 0, \quad \alpha, \beta = 1, \ldots, p, \tag{2.23}$$

where Γ_α are calculated by means of (2.10) in $\{x^i_r\}$, $R_{\alpha\beta}$ are given by (2.14), and $s \in J^p_r$ is such that $\gamma(s) \in V_{\gamma(r)}$.

Proof. For any $r \in J^p$, the restricted mapping $\gamma|_{'J^p_r} : 'J^p_r \to M$, where $'J^p_r := \{s \in J^p_r, \gamma(s) \in V_{\gamma(s)}\}$, is injective (see the above definition of J^p_r) and $\gamma|_{'J^p_r}('J^p_r) = \gamma('J^p_r)$ lies in the coordinate neighborhood $V_{\gamma(r)}$.

So, if there exists a frame $\{E'_i\}$ normal for D, then, by Proposition 2.2, Eq. (2.23) are identically satisfied.

Conversely, if (2.23) are valid, then, again, by Proposition 2.2 for every $r \in J^p$ in a certain neighborhood $'V_r$ of $\gamma('J^p_r)$ in $V_{\gamma(r)}$ exists a frame $\{E^r_i\}$ normal on $\gamma('J^p_r)$ for D_X along every vector field X. From the neighborhoods $'V_r$ we can construct a neighborhood V of $\gamma(J^p)$, e.g., by putting $V = \cup'_{r\in J^p} V_r$. Generally, V is sufficient to be taken as a union of $'V_r$ for some, but not all $r \in J^p$. On V we can obtain a normal frame $\{E'_i\}$ by putting $E'_i|_x = E^r_i|_x$ if x belongs to a single neighborhood $'V_r$. If x belongs to more than one neighborhood $'V_r$, we can choose $\{E'_i|_x\}$ to be the frame $\{E^r_i|_x\}$ for some arbitrary fixed r. □

Remark 2.2. Note that generally the frame obtained at the end of the proof of Theorem 2.1 is not continuous in the regions containing intersections of several neighborhoods $'V_r$. Hence it is, generally, no longer differentiable there. Therefore, the adjective 'normal' is not very suitable in the mentioned regions. May be in such cases is better to be spoken about 'special' frames instead of 'normal' ones.

Proposition 2.3. *If on the set $U \subseteq M$ there exists frames normal on U for some derivation along vector fields, then all of them are connected by linear transformations whose coefficients are such that the action on them of the corresponding basic vectors vanishes on U.*

Proof. If $\{E_i\}$ and $\{E'_i = A^j_i E_j\}$ are frames normal on U, i.e. if $\Gamma_X(x) = \Gamma'_X(x) = 0$ for $x \in U$ and every vector field $X = X^i E_i$, then due to (2.4), we have $X(A)|_U = 0$, i.e. $E_i(A)|_U = 0$. Conversely, if $\Gamma_X|_U = 0$ in $\{E_i\}$ and $E'_i = A^j_i E_j$ with $E_i(A)|_U = 0$, then from (2.4) follows $\Gamma'_X(x)|_U = 0$, i.e. $\{E'_i\}$ is also a normal frame. □

Proposition 2.4. *If for some derivation D along vector fields there exists a local holonomic frame normal on the set $U \subseteq M$ for D, then D is torsion-free on U. On the other hand, if D is torsion-free on U and there exist smooth (C^1) frames normal on U for D along every vector field, then all of them are holonomic on U, i.e. their basic vectors commute on U.*

Proof. If $\{E'_i\}$ is a frame normal on U, i.e. $\Gamma'_X(x) = 0$ for every X and $x \in U$, then using (2.3) and (2.6) (see also [5, Eq. (8.6)]), we find $T^D(E'_i, E'_j)|_U = -[E'_i, E'_j]|_U$. Consequently $\{E'_i\}$ is holonomic on U, i.e. $[E'_i, E'_j]|_U = 0$, iff $0 = T^D(X, Y)|_U = \{X'^i Y'^j T^D(E'_i, E'_j)\}|_U$ for every vector fields X and Y, which is equivalent to $T^D|_U = 0$.

Conversely, let $T^D|_U = 0$. We want to prove that any frame $\{E'_i = A^j_i E_j\}$ in which $\Gamma'_X = 0$ is holonomic on U. The holonomicity on U means the validity of $0 = [E'_i, E'_j]|_U = \{-(A^{-1})^l_k[E'_j(A^k_i) - E'_i(A^k_j)]E'_l\}|_U$. However (see Proposition 2.1 and (2.10)), the existence of $\{E'_i\}$ is equivalent to $\Gamma_X|_U = (\Gamma_k X^k)|_U$ for some functions Γ_k and every X. These two facts, combined with (2.3) and (2.6), lead to $(\Gamma_k)^i_j = (\Gamma_j)^i_k$. Using this and

$\{\Gamma_k A + (\partial A/\partial x^k)\}|_U = 0$ (see the proof of Proposition 2.1), we find $E'_j(A^k_i)|_U = -\{A^l_j A^m_i (\Gamma_l)^k_m\}|_U = (E'_i(A^k_j))|_U$. Therefore, $[E'_i, E'_j]|_U = 0$ (see above), i.e. $\{E'_i\}$ is holonomic on U. □

We shall end the general consideration of normal frames with the following result.

Proposition 2.5. *Let a derivation D along vector fields be linear on a set* $U \subseteq M$ *and* $\{E_i\}$ *be a frame on* $V \supseteq U$. *Then* $\{E_i\}$ *is normal on U if and only if the matrix-valued functions* Γ_k *defined via* (2.10) *in* $\{E_i\}$ *for* D_X *are such that they vanish on U,* $\Gamma_k|_U = 0$.

Proof. Let (2.10) holds in $\{E_i\}$. The frame $\{E_i\}$ is normal on U if, by definition, $\Gamma_X(x) = \Gamma_k(x)X^k(x) = 0$ in $\{E_i\}$ for $x \in U$. Since X is arbitrary, the last equation is equivalent to $\Gamma_k(x) = 0$, $x \in U$. □

Notice, if, by some reason, the vector fields are restricted somehow, e.g. if U is a submanifold and they are chosen to be tangent to it, then Proposition 2.5 may not hold (unless U is $(\dim M)$-dimensional in this example).

Consider now some special cases of the above general results.

First of all, we observe that the condition (2.13) is identically valid if $p = 0, 1$, which means that in the zero- and one-dimensional cases normal frames always exist. So, combining Theorem 2.1 and Proposition 2.1, we get the following corollary.

Corollary 2.1. *Let* $U = \{x_0\}, \gamma(J)$, *where* $x_0 \in M$ *is fixed and* $\gamma : J \to M$, *J being real interval, is a locally injective path (i.e. path with separable by neighborhoods points of self-intersection, if any). Frames normal on U for a derivation D exist if and only if D is linear on U.*

If $X|_{x_0} \neq 0$, one can always find *constant* matrices Γ_k, such that in a fixed frame $\{E_i\}$ is valid

$$\Gamma_X|_{x_0} = \Gamma_k X^k|_{x_0}, \tag{2.24}$$

which means that *any derivation along X is linear at any fixed point* x_0, *provided* $X|_{x_0} \neq 0$. Obviously, Eq. (2.24) holds for $X|_{x_0} = 0$ iff $\Gamma_X|_{x_0} = 0$ which is equivalent to $D_X(T)|_{x_0} = 0$ for any C^1 tensor field T defined on a neighborhood of x_0. Therefore, in the zero-dimensional case, the following stronger version of Corollary 2.1 is valid.

Corollary 2.2. *Let* $x_0 \in M$ *be fixed, X be a vector field and D be a derivation along vector fields. If* $X|_{x_0} \neq 0$, *then frames normal for* D_X *at* x_0 *always exist. If* $X|_{x_0} = 0$, *then frames normal for* D_X *at* x_0 *exist iff* $D_X(T)|_{x_0} = 0$ *for any* C^1 *tensor field T defined on a neighborhood of* x_0, *in which case any frame is normal for* D_X.

Of course, the (un)uniqueness and holonomicity of the normal frames in the zero- and one-dimensional case, if any, is described via Propositions 2.3 and 2.4. These results and Corollaries 2.1 and 2.2 were independently proved in [2,3] for $p = 0, 1$, respectively.

Call now attention on the other limiting case, $p = \dim M$, when U is an open neighborhood in M, possibly coinciding with the whole manifold M. In this case, the linearity of D on U means that D_X coincides on U with some covariant derivative ∇_X along X and, consequently, the r.h.s. of (2.14) represents nothing else than the components of the curvature tensor of ∇ at $\gamma(s)$ in a coordinate (holonomic) frame. Thus, combining Theorem 2.1 and Proposition 2.1, we obtain the following result, which will be proved independently because of its importance.

Corollary 2.3. *Let $U \subseteq M$ be a neighborhood and D be a derivation along vector fields. Frames normal for D on U exist if and only if D_X coincides on U with some flat linear connection ∇_X along any vector field X.*

Proof. Let us fix a frame $\{E_i\}$ in U. The existence of $\{E'_m\}$ with $\Gamma'_X = 0$, due to (2.4), implies $\Gamma_X = -(X(A))A^{-1}$, i.e. $(\Gamma_X)^i_j = -[X^k(E_k(A^i_m))](A^{-1})^m_j$ which, by (2.5), means that D coincides on U with a linear connection ∇ with local coefficients $\Gamma^i_{jk} = -(E_k(A^i_m))(A^{-1})^m_j$. Putting $\Gamma_X = -(X(A))A^{-1}$ and using $X(A^{-1}) = -A^{-1}(X(A))A^{-1}$, we get $R^D = R^\nabla = 0$. Conversely, let D coincides on U with a flat linear connection ∇. Let $\{E^0_i\}$ be a basis at $x_0 \in U$. Define the vector field E'_i so that its value $E'_i|_x$ at $x \in U$ is obtained from E^0_i by the parallel translation (transport), generated by ∇ [9,12], from x_0 to x. As ∇ is a *flat* linear connection, $E'_i|_x$ does not depend on the transportation path and the vector fields $\{E'_i\}$ are linearly independent [9,11,13], i.e. they form a frame on U. It is holonomic iff ∇ is torsion-free on U [11,13]. By definition of a parallel translation, the vectors of the frame $\{E'_i\}$ satisfy $\nabla_X E'_i = 0$, which, when combined with (2.3), implies $\Gamma'_X = 0$. □

The main consequence of Proposition 2.3 is that the (flat) linear connections are the only derivations for which normal frames exist in neighborhoods. These frames, if any, are holonomic iff the derivation is torsion-free [11,13]. From (2.4), one finds that they are connected by linear transformations with constant coefficients. By Corollary 2.3, a necessary condition for the existence of the considered special frames for a derivation D is its flatness, i.e. $R^D = 0$.

2.3. Normal frames for derivations along a fixed vector field

A derivation D_X is linear on $U \subseteq M$ along a *fixed* vector field X if (2.10) holds for $x \in U$ and the given X. In this sense, evidently, *any derivation along a fixed vector field is linear on every set* and, consequently, on the whole manifold M; moreover, generally, for a fixed X, infinitely many Γ_k, for which (2.10) holds, can be found. Namely this is the cause due to which the analogue of Proposition 2.1 for such derivations, which is evidently true, is absolutely trivial and does even need not to be formulated.

The existence of normal frames, in which the components of D_X, with a *fixed* X, vanish on some set $U \subseteq M$, significantly differs from the same problem for D_X with an *every* X (see Section 2.2). In fact, if $\{E'_i = A^j_i E_j\}$, $\{E_i\}$ being a fixed frame on U, is a frame normal on U, i.e. $\Gamma'_X|_U = 0$, then, due to (2.4), its existence is equivalent to the one of $A := [A^j_i]$ for which $(\Gamma_X A + X(A))|_U = 0$ for the *given* X. As X is *fixed*, the values of A at two different points, say $x, y \in U$, are connected through the last equation if and only if x and y lie on one and the same integral curve of X, the part of which between x and y belongs entirely to U. Hence, if $\gamma : J \to M$, J being an $\mathbb{R}$-interval, is (a part of) an integral curve of X, i.e. at $\gamma(s)$, $s \in J$ the tangent to γ vector field $\dot\gamma$ is $\dot\gamma(s) := X|_{\gamma(s)}$, then along γ the equation $(\Gamma_X A + X(A))|_U = 0$ reduces to $\mathrm{d}A/\mathrm{d}s|_{\gamma(s)} = \dot\gamma(A)|_s = (X(A))|_{\gamma(s)} = -\Gamma_X(\gamma(s))A(\gamma(s))$. Applying Lemma 2.1 for $p = 1$, we see that the general solution of this equation is

$$A(s;\gamma) = Y(s, s_0; -\Gamma_X \circ \gamma)B(\gamma), \tag{2.25}$$

where $s_0 \in J$ is fixed, $Y = Y(s, s_0; Z)$, with Z being a C^1 matrix function of s, is the unique solution of the initial-value problem (see [14, Ch. IV, Section 1])

$$\frac{\mathrm{d}Y}{\mathrm{d}s} = ZY, \quad Y|_{s=s_0} = \mathbb{1}, \tag{2.26}$$

and the non-degenerate matrix $B(\gamma)$ may depend only on γ, but not on s. (Note that (2.26) is a special case of (2.11) for $p = 1$ and by Lemma 2.1 it has always a unique solution because $R_{11}(Z_1) \equiv 0$, due to (2.12) with $p = 1$.)

From the above considerations, the next propositions follow.

Proposition 2.6. *There exist frames normal for any derivation along a fixed vector field on every set $U \subseteq M$.*

Proposition 2.7. *The frames normal on a set $U \subseteq M$ for some derivation along a fixed vector field X are connected by linear transformations whose matrices are such that the action of X on them vanishes on U.*

Proof. If $\{E_i\}$ and $\{E'_i = A^j_i E_j\}$ are such that $\Gamma'_X|_U = \Gamma_X|_U = 0$, then, due to (2.4), we have $X(A)|_U = 0$. On the other hand, if $\Gamma_X|_U = 0$ and $X(A)|_U = 0$, then, by (2.4), is fulfilled $\Gamma'_X|_U = 0$, i.e. $\{E'_i\}$ is normal. □

The problem for the holonomicity of frames normal for a derivation D_X along a fixed vector field X is, generally, ill-posed. In fact, on one hand, the concept of torsion for such a derivation is not defined (see (2.6), where X and Y are arbitrary vector fields) and, on other hand, to talk about holonomicity of a frame it is necessary it to be defined on a neighborhood in a smooth way, while the frames normal for D_X on $U \subseteq M$ are, generally, such only along the integral paths of X lying in U. An exception of this conclusion is the case $U = \gamma(J)$ with $\gamma : J \to M$ being an integral path of X. In it there are holonomic as well anholonomic frames defined *on a neighborhood of* $\gamma(J)$ and *normal on* $\gamma(J)$ (see Lemma 2.2).

2.4. Normal frames for derivations along paths

Let $\gamma : J \to M$, J being an $\mathbb{R}$-interval, be a C^1 injective path and X be a C^1 vector field defined on a neighborhood of $\gamma(J)$ in such a way that on $\gamma(J)$ it reduces to the tangent vector field $\dot\gamma$, i.e. $X|_{\gamma(s)} = \dot\gamma(s)$, $s \in J$. We call the restriction on $\gamma(J)$ of a derivation D_X along X *derivation along* γ and denote it by $\mathcal{D}^\gamma$. Of course, $\mathcal{D}^\gamma$ generally depends on the values of X outside $\gamma(J)$, but, as this dependence is insignificant for the following, it will not be written explicitly. So, if T is a C^1 tensor field in a neighborhood of $\gamma(J)$, then[1]

$$(\mathcal{D}^\gamma T)(\gamma(s)) := \mathcal{D}^\gamma_s T := (D_X T)|_{\gamma(s)}, \quad X|_{\gamma(s)} = \dot\gamma(s). \tag{2.27}$$

If D is a derivation along vector fields *linear* on $\gamma(J)$, it is easily seen (see (2.1) and (2.10)) that $\mathcal{D}^\gamma_s T$ depends only on the values of $T|_x$ for $x \in \gamma(J)$, but not on the ones for $x \notin \gamma(J)$. The operator $\mathcal{D}^\gamma$ is a generalization of the usual covariant differentiation along curves (see [11,15,16]).

When restricted to $\gamma(J)$, the components of D_X will be called components of $\mathcal{D}^\gamma$. Since we can regard X in (2.27) as a fixed vector field, the next proposition is a consequence of Proposition 2.6, but we shall present bellow its independent proof too.

Proposition 2.8. *Along any C^1 injective path $\gamma : J \to M$ there exists a frame $\{E'_i\}$ in which the components of a given derivation $\mathcal{D}^\gamma$ along γ vanish on $\gamma(J)$.*

Proof. Let us fix a frame $\{E_i\}$ in a neighborhood of $\gamma(J)$. We have to prove the existence of a transformation $\{E_i\} \to \{E'_j = A^i_j E_i\}$ such that $\Gamma'_X|_{\gamma(J)} = 0$. By (2.4), this is equivalent to the existence of a matrix function $A = [A^i_j]$ satisfying along γ the equation $(A^{-1}(\Gamma_X A + X(A)))|_{\gamma(J)} = 0$, $s \in J$, or

$$\dot\gamma(A)|_{\gamma(s)} \equiv \frac{\mathrm{d}A(\gamma(s))}{\mathrm{d}s} = -\Gamma_X(\gamma(s))A(\gamma(s)) \tag{2.28}$$

as $X|_{\gamma(s)} = \dot\gamma(s)$. The general solution of this equation with respect to A is

$$\mathbf{A}(s;\gamma) = Y(s, s_0; -\Gamma_X \circ \gamma)B(\gamma), \tag{2.29}$$

where Y is the unique solution of the initial-value problem (2.11) with $p = 1$, $s_0 \in J$ is fixed, and $B(\gamma)$ is a non-degenerate matrix function of γ.

Let A be any matrix function with the property $A(x)|_{x=\gamma(s)} = \mathbf{A}(s;\gamma)$ for some s_0 and B (e.g., using the notation of the proof of Proposition 2.2 with $p = 1$, in any coordinate neighborhood in which γ is without self-intersections, we can put $A(\eta(s,\mathbf{t})) = Y(s, s_0; -\Gamma_X \circ \gamma)B(s_0, \mathbf{t}_0, \mathbf{t}; \gamma)$ for a fixed non-degenerate matrix function B.) Then it is easily seen that A carries out the needed transformation. Hence, the frame $\{E'_j = A^i_j E_i\}$ is the one looked for. □

The frames provided by Proposition 2.8 will be called *normal* for $\mathcal{D}^\gamma$ (along γ).

Proposition 2.9. *The frames normal along an injective path $\gamma : J \to M$ for $\mathcal{D}^\gamma$ are connected by linear transformations whose coefficients on $\gamma(J)$ are constant or may depend only on γ.*

[1] Here and below the condition γ to be injective is essential one as otherwise the mapping $\gamma(s) \mapsto \dot\gamma(s)$ is not a (single-valued) vector field at the points of self-intersection of γ. The theory below can be generalized for an arbitrary, injective or not such, path γ if by $X(f)|_{\gamma(s)}$, f being a C^1 function, one understands $\mathrm{d}f(\gamma(s))/\mathrm{d}s$.

Proof. If $\{E_i\}$ and $\{E'_i\}$ are normal frames, then $\Gamma_X(\gamma(s)) = \Gamma'_X(\gamma(s)) = 0$, $X|_{\gamma(s)} = \dot\gamma(s)$. So, from (2.4) follows $\dot\gamma(A)|_{\gamma(s)} = \mathrm{d}A(\gamma(s))/\mathrm{d}s = 0$, i.e. $A(\gamma(s))$ is a constant or depends only on the mapping γ. □

From Propositions 2.8 and 2.9, we infer that the requirement for the components of $\mathcal{D}^\gamma$ to vanish along a path γ determines the corresponding normal frames with some arbitrariness only on $\gamma(J)$ and leaves them absolutely arbitrary outside the set $\gamma(J)$. For this reason, we speak about frames normal for $\mathcal{D}^\gamma$ defined only *on* $\gamma(J)$.

Proposition 2.10. *Let the frame $\{E'_i\}$ defined on $\gamma(J)$ be normal for some derivation $\mathcal{D}^\gamma$ along a C^1 injective path $\gamma : J \to M$. Let U be a coordinate neighborhood. Then there is a neighborhood of $U \cap (\gamma(J))$ in U in which $\{E'_i\}$ can be extended to a coordinate frame, i.e. in this neighborhood there exist local coordinates $\{y^i\}$ such that $E'_i|_{\gamma(s)} = \partial/\partial y^i|_{\gamma(s)}$.*

Remark 2.3. This proposition means that locally any frame normal for $\mathcal{D}^\gamma$ on $\gamma(J)$ can be thought of as (extended to) a coordinate, and hence holonomic, one (see Proposition 2.9). In particular, if γ is contained in only one coordinate neighborhood and is injective, as it is supposed, then every frame normal on $\gamma(J)$ for $\mathcal{D}^\gamma$ can be extended to a holonomic one (see the proof of Proposition 2.9).

Remark 2.4. This result is independent of the torsion of the derivation D which induces $\mathcal{D}^\gamma$. The cause for this is the condition $X|_{\gamma(s)} = \dot\gamma(s)$ in (2.27).

Proof. The proposition is a trivial corollary from the proof of Proposition 2.8 and the following lemma. □

Lemma 2.2. *Let the path $\gamma : J \to M$ be without self-intersections and such that $\gamma(J)$ is contained in some coordinate neighborhood U,* i.e. *$\gamma(J) \subset U$. Let $\{E'_i\}$ be a smooth frame defined on $\gamma(J)$,* i.e. *$E'_i|_{\gamma(s)}$ depends smoothly on s. Then there is a neighborhood of $\gamma(J)$ in U in which coordinates $\{y^i\}$ exist such that $E'_i|_{\gamma(s)} = \partial/\partial y^i|_{\gamma(s)}$,* i.e. *$\{E'_i\}$ can be extended in it to a coordinate frame.*

Proof. Let $\eta : J \times V \to U$, $V := J \times \cdots \times J$ $(n-1$ times), be a C^1 one-to-one onto mapping such that $\eta(\cdot, \mathbf{t}_0) = \gamma$ for some fixed $\mathbf{t}_0 \in V$, i.e. $\eta(s, \mathbf{t}_0) = \gamma(s)$, $s \in J$ (cf. the proof of Proposition 2.2). In the neighborhood $\eta(J, V) \subset U$ we introduce coordinates $\{x^i\}$ by putting $(x^1(\eta(s,\mathbf{t})), \ldots, x^n(\eta(s,\mathbf{t}))) = (s, \mathbf{t})$, $s \in J$, $\mathbf{t} \in V$. Let the non-degenerate matrix $[A^i_j(s;\gamma)]$ defines the expansion of $\{E'_i\}$ with respect to $\{\partial/\partial x^i\}$, i.e.

$$E'_i|_{\gamma(s)} = A^j_i(s;\gamma)\left(\left.\frac{\partial}{\partial x^j}\right|_{\gamma(s)}\right). \tag{2.30}$$

Define the functions $y^i : \eta(J, V) \to \mathbb{R}$ by

$$y^i(\eta(s,\mathbf{t})) := x^i_0 + \int_{s_0}^{s} (A^{-1})^i_1(u;\gamma)\,\mathrm{d}u + (A^{-1})^i_j(s;\gamma)[x^j(\eta(s,\mathbf{t})) - x^j(\gamma(s))] + f^i_{jk}(s,\mathbf{t};\gamma)[x^j(\eta(s,\mathbf{t})) - x^j(\gamma(s))][x^k(\eta(s,\mathbf{t})) - x^k(\gamma(s))], \tag{2.31}$$

where $s_0 \in J$ and $x_0 \in \eta(J, V)$ are fixed and the functions f^i_{jk} together with their first derivatives are bounded when $\mathbf{t} \to \mathbf{t}_0$. Then, because of $\eta(\cdot, \mathbf{t}_0) = \gamma$, we find

$$\frac{\partial y^i}{\partial x^j}\bigg|_{\gamma(s)} = \frac{\partial y^i}{\partial x^j}\bigg|_{\eta(s,\mathbf{t}_0)} = (A^{-1})^i_j(s;\gamma). \tag{2.32}$$

As $\det[A^i_j(s;\gamma)] \neq 0, \infty$, from (2.32) it follows that the transformation $\{x^i\} \to \{y^i\}$ is a diffeomorphism on some neighborhood of $\gamma(J)$ lying in U. So, in this neighborhood $\{y^i\}$ are local coordinates. The coordinate basic vectors on $\gamma(J)$ corresponding to them are (see (2.30) and (2.32))

$$\frac{\partial}{\partial y^j}\bigg|_{\gamma(s)} = \left(\frac{\partial x^i}{\partial y^j}\bigg|_{\gamma(s)}\right) \frac{\partial}{\partial x^i}\bigg|_{\gamma(s)} = A^i_j(s;\gamma)\frac{\partial}{\partial x^i}\bigg|_{\gamma(s)} = E'_j|_{\gamma(s)}.$$

Hence $\{y^i\}$ are the local coordinates we are looking for. □

Lemma 2.2 has also a separate meaning: according to it any locally smooth frame defined *on* $\gamma(J)$ can locally be extended to a *holonomic* frame *in a neighborhood of* $\gamma(J)$. Evidently, such an extension can be done in an anholonomic way too. Consequently, the holonomicity problem for a frame defined only on $\gamma(J)$ depends on the way this frame is extended in a neighborhood of $\gamma(J)$.

2.5. Normal frames for linear connections on a manifold

As it was said in Section 2.2, the linear connections (covariant derivatives) on a manifold M are derivations of the tensor algebra over it which are linear on any subset $U \subseteq M$. For a linear connection ∇ the matrices $\Gamma_k(x)$ in (2.10) are nothing else but the ones formed from the local coefficients $\Gamma^i_{jk}(x)$ of ∇ in a frame $\{E_i\}$ (see (2.5)), $\Gamma_k(x) = [\Gamma^i_{jk}(x)]^{\dim M}_{i,j=1}$. Therefore, according to Proposition 2.5, a frame is normal for ∇ on U iff in it the coefficients of ∇ vanish on U. This simple conclusion completely agrees with the definition of a frame normal for a linear connection given in the literature [1,11,12,16,17] as a one in which the connection's coefficients vanish (on some set).

Because of the importance of the linear connections on a manifold in geometry and theoretical physics, below we present a partial list of some results regarding frames normal for them.

Corollary 2.4. *Let the points of self-intersection of the C^1 mapping $\gamma : J^p \to M$, if any, be separable by neighborhoods, ∇ be a linear connection on M with local coefficients Γ^i_{jk} (in a frame $\{E_i\}$) and $\Gamma_k := [\Gamma^i_{jk}]^n_{i,j=1}$. Then in a neighborhood of $\gamma(J^p)$ there exists a frame $\{E'_i\}$ normal on $\gamma(J^p)$ for ∇, i.e. $\Gamma'_k|_{\gamma(J^p)} = 0$, iff for every $r \in J^p$ in the coordinates $\{x^i_r\}$ (defined before Theorem 2.1) is satisfied (2.13) in which Γ_α, $\alpha = 1, \dots, p$ are part of the components of ∇ in $\{x^i_r\}$ and $s \in J^p$ is such that $\gamma(s) \in V_{\gamma(r)}$.*

Proof. For linear connections, Eq. (2.10) is valid for every X in any frame. So, if in a frame $\{E'_i\}$ is fulfilled $\Gamma'_X|_U = 0$ for $U \subseteq M$, we have in it $\Gamma'_k|_U = 0$ (see (2.4)) and vice versa, if in a frame $\{E'_i\}$ is valid $\Gamma'_k|_U = 0$, then $\Gamma_X|_U = 0$ for every X. Combining this fact with Theorem 2.1, we get the required result. □

Corollary 2.5. *If on a set $U \subseteq M$, there exist frames normal for some linear connection on U, then these frames are connected by linear transformations whose matrices are such that the action of the corresponding basic vectors on them vanishes on U.*

Proof. The result follows from Proposition 2.3 and the proof of Corollary 2.4. □

Corollary 2.6. *Let there exist locally smooth frames normal on U for some linear connection on a neighborhood of some set $U \subseteq M$. Then one* (*and hence any*) *such frame is holonomic on U iff the connection is torsion-free on U.*

Proof. The statement follows from (2.10) (or (2.5)) and Proposition 2.4. □

Corollary 2.7. *For any linear connection ∇, there exist frames along every path $\gamma : J \to M$ in which the coefficients of ∇ vanish on $\gamma(J)$. These normal frames are connected with one another in the way described in* Proposition 2.3.

Proof. This result is a consequence from (2.5), Propositions 2.1 and 2.3 and their proofs; in the former of the proofs a frame with the needed property is explicitly constructed. □

Corollary 2.8. *One, and hence any, frame for a linear connection ∇, which is smooth on $\gamma(J)$ and normal along a path $\gamma : J \to M$, is holonomic if and only if ∇ is torsion-free on $\gamma(J)$.*

Proof. The statement follows from (2.5) and Propositions 2.1 and 2.4. □

Corollary 2.9. *Let ∇ be a torsion-free linear connection and the path $\gamma : J \to M$ be without self-intersections and lying in only one coordinate neighborhood. Then for ∇ there exist coordinates normal on $\gamma(J)$, or, equivalently, locally holonomic normal frames.*

Proof. The result follows from Corollaries 2.7 and 2.8. □

Corollary 2.10. *Let $D/\mathrm{d}s|_\gamma := \nabla_{\dot\gamma}$ be the covariant derivative associated with a linear connection ∇ along some C^1 injective path $\gamma : J \to M$. Then on $\gamma(J)$ there exist frames normal for $\nabla_{\dot\gamma}$. They are obtained from one another by linear transformations whose coefficients are constant or depend only on γ. If $\gamma(J)$ lies in a single coordinate neighborhood, then in some neighborhood of $\gamma(J)$ all of these normal frames can be extended in a holonomic way.*

Proof. The statement follows from Propositions 2.8–2.10. □

3. Normal frames for linear connections in vector bundles and the equivalence principle

The main purpose of the present, third, part of our work is problems similar to those in Section 2 to be investigated for linear connections in finite-dimensional vector bundles. It will be demonstrated that the results obtained in Section 2 can be almost automatically reformulated to solve these problems. Besides, some speculation concerning the equivalence principle in classical gauge theories will be presented.

3.1. Linear connections in vector bundles

Different equivalent definitions of a (linear) connection in vector bundles are known and in current usage [9,18–20]. The most suitable one for our purposes is given in [21, p. 223] (see also [18, Theorem 2.52]).

Suppose (E, π, M), E and M being finite-dimensional C^∞ manifolds, be C^∞ $\mathbb{K}$-vector bundle [18] with bundle space E, base M, and projection $\pi : E \to M$. Here $\mathbb{K}$ stands for the field $\mathbb{R}$ of real numbers or $\mathbb{C}$ of complex ones. Let $\mathrm{Sec}^k(E, \pi, M)$, $k = 0, 1, 2, \dots$ be the set (in fact the module) of C^k sections of (E, π, M) and $\mathcal{X}(M)$ the one of vector fields on M.

Definition 3.1. Let $V, W \in \mathcal{X}(M)$, $\sigma, \tau \in \mathrm{Sec}^1(E, \pi, M)$, and $f : M \to \mathbb{K}$ be a C^∞ function. A mapping $\nabla : \mathcal{X}(M) \times \mathrm{Sec}^1(E, \pi, M) \to \mathrm{Sec}^0(E, \pi, M)$, $\nabla : (V, \sigma) \mapsto \nabla_V \sigma$, is called a (*linear*) *connection in* (E, π, M) if

$$\nabla_{V+W}\sigma = \nabla_V\sigma + \nabla_W\sigma, \tag{3.1a}$$

$$\nabla_{fV}\sigma = f\nabla_V\sigma, \tag{3.1b}$$

$$\nabla_V(\sigma + \tau) = \nabla_V(\sigma) + \nabla_V(\tau), \tag{3.1c}$$

$$\nabla_V(f\sigma) = V(f)\cdot\sigma + f\cdot\nabla_V(\sigma). \tag{3.1d}$$

Remark 3.1. Rigorously speaking, ∇, as defined by Definition 3.1, is a covariant derivative operator in (E, π, M)—see [18, Definition 2.51]—but, as a consequence of [18, Theorem 2.52], this cannot lead to some ambiguities.

Remark 3.2. Since $V(a) = 0$ for every $a \in \mathbb{K}$ (considered as a constant function $M \to \{a\}$), the mapping $\nabla : (V, \sigma) \mapsto \nabla_V\sigma$ is $\mathbb{K}$-linear with respect to both its arguments.

Let $\{e_i : i = 1, \dots, \dim \pi^{-1}(x)\}$, $x \in M$ and $\{E_\mu : \mu = 1, \dots, \dim M\}$ be frames over an open set $U \subseteq M$ in, respectively (E, π, M) and the tangent bundle $(T(M), \pi_T, M)$ over M, i.e. for every $x \in U$, the set $\{e_i|_x\}$ forms a basis of the fiber $\pi^{-1}(x)$ and $\{E_\mu|_x\}$ is a basis of the space $T_x(M) = \pi_T^{-1}(x)$ tangent to M at x. Let us write $\sigma = \sigma^i e_i$ and $V = V^\mu E_\mu$, where here and henceforth the Latin (resp. Greek) indices run from 1 to the dimension of (E, π, M) (resp. M), the Einstein summation convention is assumed, and $\sigma^i, V^\mu : U \to \mathbb{K}$ are some C^1 functions. Then, from Definition 3.1, one gets

$$\nabla_V \sigma = V^\mu (E_\mu(\sigma^i) + \Gamma^i_{j\mu}\sigma^j) e_i, \tag{3.2}$$

where $\Gamma^i_{j\mu} : U \to \mathbb{K}$, called *coefficients* of ∇, are given by

$$\nabla_{E_\mu} e_j =: \Gamma^i_{j\mu} e_i. \tag{3.3}$$

Evidently, by virtue of (3.2), the knowledge of $\{\Gamma^i_{j\mu}\}$ in a pair of frames $(\{e_i\}, \{E_\mu\})$ over U is equivalent to the one of ∇ as any transformation $(\{e_i\}, \{E_\mu\}) \mapsto (\{e'_i = A^j_i e_j\}, \{E'_\mu = B^\nu_\mu E_\nu\})$ with non-degenerate matrix-valued functions $A = [A^j_i]$ and $B = [B^\nu_\mu]$ on U implies $\Gamma^i_{j\mu} \mapsto \Gamma^{i'}_{j\mu}$ with

$$\Gamma'^i_{j\mu} = \sum_{\nu=1}^{\dim M} \sum_{k,l=1}^{\dim \pi^{-1}(x)} B^\nu_\mu (A^{-1})^i_k A^l_j \Gamma^k_{l\nu} + \sum_{\nu=1}^{\dim M} \sum_{k=1}^{\dim \pi^{-1}(x)} B^\nu_\mu (A^{-1})^i_k E_\nu(A^k_j) \tag{3.4}$$

which in a matrix form reads

$$\Gamma'_\mu = B^\nu_\mu A^{-1} \Gamma_\nu A + A^{-1} E'_\mu(A) = B^\nu_\mu A^{-1}(\Gamma_\nu A + E_\nu(A)), \tag{3.5}$$

where $\Gamma_\mu := [\Gamma^i_{j\mu}]^{\dim \pi^{-1}(x)}_{i,j=1}$, $x \in M$, and $\Gamma'_\mu := [\Gamma'_{j\mu}{}^i]^{\dim \pi^{-1}(x)}_{i,j=1}$.

The interpretation of the coefficients $\Gamma^i_{j\mu}$ as components of a 1-form (more precisely, of endomorphisms of E-valued 1-form or section of the endomorphism bundle of (E, π, M), or of Lie algebra-valued 1-form in a case of principle bundle) is well known and considered at length in the literature [10,18,20–22], but it will not be needed directly in the present work.

3.2. *Linear transports along paths in vector bundles*

To begin with, we recall some definitions and results from the paper [8].[2] Below we denote by $\mathrm{PLift}^k(E, \pi, M)$ the set of liftings of C^k paths from M to E such that the lifted paths are of class C^k, $k = 0, 1, \ldots$. Let $\gamma : J \to M$, J being real interval, be a path in M.

Definition 3.2. A *linear transport along paths in vector bundle* (E, π, M) is a mapping L assigning to every path γ a mapping L^γ, *transport along* γ, such that $L^\gamma : (s, t) \mapsto L^\gamma_{s\to t}$ where the mapping

$$L^\gamma_{s\to t} : \pi^{-1}(\gamma(s)) \to \pi^{-1}(\gamma(t)) \quad , s, t \in J, \tag{3.6}$$

called *transport along* γ *from* s *to* t, has the properties:

$$L^\gamma_{s\to t} \circ L^\gamma_{r\to s} = L^\gamma_{r\to t}, \quad r, s, t \in J, \tag{3.7}$$

$$L^\gamma_{s\to s} = \mathrm{id}_{\pi^{-1}(\gamma(s))}, \quad s \in J, \tag{3.8}$$

[2] In [8] is assumed $\mathbb{K} = \mathbb{C}$, but this choice is insignificant.

$$L^{\gamma}_{s\to t}(\lambda u+\mu v)=\lambda L^{\gamma}_{s\to t}u+\mu L^{\gamma}_{s\to t}v,\quad \lambda,\mu\in\mathbb{K},\quad u,v\in\pi^{-1}(\gamma(s)),\tag{3.9}$$

where $\circ$ denotes composition of maps and id_X is the identity map of a set X.

Definition 3.3. A *derivation along paths in* (E,π,M) or a *derivation of liftings of paths in* (E,π,M) is a mapping

$$D:\mathrm{PLift}^1(E,\pi,M)\to\mathrm{PLift}^0(E,\pi,M)\tag{3.10a}$$

which is $\mathbb{K}$-linear,

$$D(a\lambda+b\mu)=aD(\lambda)+bD(\mu)\tag{3.11a}$$

for $a,b\in\mathbb{K}$ and $\lambda,\mu\in\mathrm{PLift}^1(E,\pi,M)$, and the mapping

$$D^{\gamma}_s:\mathrm{PLift}^1(E,\pi,M)\to\pi^{-1}(\gamma(s)),\tag{3.10b}$$

defined via $D^{\gamma}_s(\lambda):=((D(\lambda))(\gamma))(s)=(D\lambda)_{\gamma}(s)$ and called *derivation along* $\gamma:J\to M$ *at* $s\in J$, satisfies the 'Leibniz rule':

$$D^{\gamma}_s(f\lambda)=\frac{\mathrm{d}f_{\gamma}(s)}{\mathrm{d}s}\lambda_{\gamma}(s)+f_{\gamma}(s)D^{\gamma}_s(\lambda)\tag{3.11b}$$

for every

$$f\in\mathrm{PF}^1(M):=\{\varphi|\varphi:\gamma\mapsto\varphi_{\gamma},\gamma:J\to M,\varphi_{\gamma}:J\to\mathbb{K}\ \text{being of class}\ C^1\}.$$

The mapping

$$D^{\gamma}:\mathrm{PLift}^1(E,\pi,M)\to P(\pi^{-1}(\gamma(J))):=\{\text{paths in}\ \pi^{-1}(\gamma(J))\},\tag{3.10c}$$

defined by $D^{\gamma}(\lambda):=(D(\lambda))|_{\gamma}=(D\lambda)_{\gamma}$, is called *derivation along* γ.

If $\gamma:J\to M$ is a path in M and $\{e_i(s;\gamma)\}$ is a basis in $\pi^{-1}(\gamma(s))$,[3] in the frame $\{e_i\}$ over $\gamma(J)$ the *components* (*matrix elements*)$L^i_j:U\to\mathbb{K}$ of a linear transport L along paths and the ones of a derivation D along paths in vector bundle (E,π,M) are defined through, respectively,

$$L^{\gamma}_{s\to t}(e_i(s;\gamma))=:L^j_i(t,s;\gamma)e_j(t;\gamma)\quad ,s,t\in J,\tag{3.12}$$

$$D^{\gamma}_s\hat{e}_j=:\Gamma^i_j(s;\gamma)e_i(s;\gamma),\quad s\in J,\tag{3.13}$$

where $\hat{e}_i:\gamma\mapsto e_i(\cdot;\gamma)$ is a lifting of γ generated by e_i.

It is a simple exercise to verify that the components of L and D uniquely define (locally) their action on $u=u^ie_i(s;\gamma)$ and $\lambda\in\mathrm{PLift}^1(E,\pi,M)$, $\lambda:\gamma\mapsto\lambda_{\gamma}=\lambda^i_{\gamma}\hat{e}_i$, according to

$$L^{\gamma}_{s\to t}u=:L^i_j(t,s;\gamma)u^je_i(t;\gamma),\tag{3.14}$$

$$D^{\gamma}_s\lambda=:\left(\frac{\mathrm{d}\lambda^i_{\gamma}(s)}{\mathrm{d}s}+\Gamma^i_j(s;\gamma)\lambda^j_{\gamma}(s)\right)e_i(s;\gamma)\tag{3.15}$$

[3] If there are $s_1,s_2\in J$ such that $\gamma(s_1)=\gamma(s_2):=y$, the vectors $e_i(s_1;\gamma)$ and $e_i(s_2;\gamma)$ need not coincide. So, if this is the case, the bases $\{e_i(s_1;\gamma)\}$ and $\{e_i(s_2;\gamma)\}$ in $\pi^{-1}(y)$ may turn to be different.

and that a change $\{e_i(s;\gamma)\} \mapsto \{e'_i(s;\gamma) = A_i^j(s;\gamma)e_j(s;\gamma)\}$, with a non-degenerate matrix-valued function $A(s;\gamma) := [A_i^j(s;\gamma)]$, implies the transformation

$$\boldsymbol{L}(t,s;\gamma) := [L_i^j(t,s;\gamma)] \mapsto \boldsymbol{L}'(t,s;\gamma) = A^{-1}(t;\gamma)\boldsymbol{L}(t,s;\gamma)A(s;\gamma), \tag{3.16}$$

$$\begin{aligned}\mathbf{\Gamma}(s;\gamma) &:= [\Gamma_j^i(s;\gamma)] \mapsto \mathbf{\Gamma}'(s;\gamma) \\ &= A^{-1}(s;\gamma)\mathbf{\Gamma}(s;\gamma)A(s;\gamma) + A^{-1}(s;\gamma)\frac{\mathrm{d}A(s;\gamma)}{\mathrm{d}s}.\end{aligned} \tag{3.17}$$

A crucial role further will be played by the *coefficients* $\Gamma_j^i(s;\gamma)$ in a frame $\{e_i\}$ of linear transport L,

$$\Gamma_j^i(s;\gamma) := \left.\frac{\partial L_j^i(s,t;\gamma)}{\partial t}\right|_{t=s} = -\left.\frac{\partial L_j^i(s,t;\gamma)}{\partial s}\right|_{t=s}. \tag{3.18}$$

The usage of the same notation for the *coefficients* of a transport L and *components* of derivation D along paths is not accidental and finds its reason in the next fundamental result [8, Section 2]. Call a transport L differentiable of class C^k, $k = 0, 1, \ldots$ if its matrix $\boldsymbol{L}(t,s;\gamma)$ has C^k dependence on t (and hence on s—see [8, Section 2]). *Every C^1 linear transport L along paths generates a derivation D along paths via*

$$D_s^\gamma(\lambda) := \lim_{\varepsilon\to 0}\left\{\frac{1}{\varepsilon}[L_{s+\varepsilon\to s}^\gamma\lambda_\gamma(s+\varepsilon) - \lambda_\gamma(s)]\right\} \tag{3.19}$$

for every lifting $\lambda \in \mathrm{PLift}^1(E,\pi,M)$ with $\lambda : \gamma \mapsto \lambda_\gamma$ and conversely, for any derivation D along paths there exists a unique linear transport along paths generating it via (3.19). Besides, if L and D are connected via (3.19), the coefficients of L coincide with the components of D. In short, there is a bijective correspondence between linear transports and derivations along paths given locally through the equality of their coefficients and components, respectively.

More details and results on the above items can be found in [8].

3.3. Links between linear connections and linear transports

Suppose $\gamma : J \to M$ is a C^1 path and $\dot\gamma(s)$, $s \in J$, is the vector tangent to γ at $\gamma(s)$ (more precisely, at s). Let ∇ and D be, respectively, a linear connection and derivation along paths in vector bundle (E,π,M) and in a pair of frames $(\{e_i\},\{E_\mu\})$ over some open set in M the coefficients of ∇ and the components of D be $\Gamma_{j\mu}^i$ and Γ_j^i, respectively, i.e. $\nabla_{E_\mu} = \Gamma_{i\mu}^j e_j$ and $D_s^\gamma \hat{e}_i = \Gamma_i^j e_j(\gamma(s))$ with $\hat{e}_i : \gamma \mapsto \hat{e}_i|_\gamma : s \mapsto e_i(\gamma(s))$ being lifting of paths generated by e_i. If $\sigma = \sigma^i e_i \in \mathrm{Sec}^1(E,\pi,M)$ and $\hat\sigma \in \mathrm{PLift}(E,\pi,M)$ is given via $\hat\sigma : \gamma \mapsto \hat\sigma_\gamma := \sigma\circ\gamma$, then (3.15) implies

$$D_s^\gamma\hat\sigma = \left(\frac{\mathrm{d}\sigma^i(\gamma(s))}{\mathrm{d}s} + \Gamma_j^i(s;\gamma)\sigma^j(\gamma(s))\right)e_i(\gamma(s)),$$

while if $\gamma(s)$ is not a self-intersection point for γ, Eq. (3.2) leads to

$$(\nabla_{\dot\gamma}\sigma)|_{\gamma(s)} = \left(\frac{\mathrm{d}\sigma^i(\gamma(s))}{\mathrm{d}s} + \Gamma^i_{j\mu}(\gamma(s))\sigma^j(\gamma(s))\dot\gamma^\mu(s)\right) e_i(\gamma(s)).$$

Obviously, we have

$$D^\gamma_s\hat\sigma = (\nabla_{\dot\gamma}\sigma)|_{\gamma(s)} \tag{3.20}$$

for every σ iff

$$\Gamma^i_j(s;\gamma) = \Gamma^i_{j\mu}(\gamma(s))\dot\gamma^\mu(s), \tag{3.21}$$

which in matrix form reads

$$\mathbf{\Gamma}(s;\gamma) = \Gamma_\mu(\gamma(s))\dot\gamma^\mu(s). \tag{3.21}$$

A simple algebraic calculation shows that this equality is invariant under changes of the frames $\{e_i\}$ in (E,π,M) and $\{E_\mu\}$ in $(T(M),\pi_T,M)$. Besides, if (3.21) holds, then $\mathbf{\Gamma}$ transforms according to (3.17) iff Γ_μ transforms according to (3.5).

The above considerations are a hint that the linear connections should, and in fact can, be described in terms of derivations or, equivalently, linear transports along paths; the second description being more relevant if one is interested in the parallel transports generated by connections.

Theorem 3.1. *If ∇ is a linear connection, then there exists a derivation D along paths such that (3.20) holds for every C^1 path $\gamma : J \to M$ and every $s \in J$ for which $\gamma(s)$ is not self-intersection point for γ.*[4] *The matrix of the components of D is given by (3.21) for every C^1* path *$\gamma : J \to M$ and $s \in J$ such that $\gamma(s)$ is not a self-intersection point for γ. Conversely, given a derivation D along path whose matrix along any C^1 path $\gamma : J \to M$ has the form (3.21) for some matrix-valued functions Γ_μ, there is a unique linear connection ∇ whose matrices of coefficients are exactly Γ_μ and for which, consequently (3.20) is valid at the not self-intersection points of γ.*

Proof. NECESSITY. If Γ_μ are the matrices of the coefficients of ∇ in some pair of frames $(\{e_i\},\{E_\mu\})$, define the matrix $\mathbf{\Gamma}$ of the components of D via (3.21) for any $\gamma : J \to M$. SUFFICIENCY. Given D for which the decomposition (3.21) holds in $(\{e_i\},\{E_\mu\})$ for any γ. It is trivial to verity that Γ_μ transform according to (3.5) and, consequently, they are the matrices of the coefficients of a linear connection ∇ for which, evidently, (3.20) holds. □

A trivial consequence of the above theorem is the next important result.

Corollary 3.1. *There is a bijective correspondence between the set of linear connections in a vector bundle and the one of derivations along paths in it whose components' matrices*

[4] In particular, γ can be injective and s arbitrary. If we restrict the considerations to injective paths, the derivation D is unique. The essential point here is that at the self-intersection points of γ, if any, the mapping $\dot\gamma : \gamma(s) \mapsto \dot\gamma(s)$ is generally multiple-valued and, consequently, it is not a vector field (along γ); as a result $(\nabla_{\dot\gamma}\sigma)|_{\gamma(s)}$ at them becomes also multiple-valued.

admit (locally) the decomposition (3.21). Locally, along a C^1 path γ and pair of frames $(\{e_i\}, \{E_\mu\})$ along it, it is given by (3.21) in which $\mathbf{\Gamma}$ and Γ_μ are the matrices of the components of a derivation along paths and of the coefficients of a linear connection, respectively.

Let us now look on the preceding material from the view-point of linear transports along paths and parallel transports generated by linear connections.

Recall (see, e.g. [18, Chapter 2]), a section $\sigma \in \mathrm{Sec}^1(E, \pi, M)$ is *parallel along C^1* path $\gamma : J \to M$ with respect to a linear connection ∇ if $(\nabla_{\dot\gamma}\sigma)|_{\gamma(s)} = 0$, $s \in J$.[5] The *parallel transport along* a C^1 path $\alpha : [a, b] \to M$, $a, b \in \mathbb{R}$, $a \le b$, generated by ∇ is a mapping

$$P^\alpha : \pi^{-1}(\alpha(a)) \to \pi^{-1}(\alpha(b))$$

such that $P^\alpha(u_0) := u(b)$ for every element $u_0 \in \pi^{-1}(\alpha(a))$, where $u \in \mathrm{Sec}^1(E, \pi, M)|_{\alpha([a,b])}$ is the unique solution of the initial-value problem

$$\nabla_{\dot\alpha} u = 0, \qquad u(a) = u_0. \tag{3.23}$$

The *parallel transport* P generated by (assigned to, corresponding to) a linear connection ∇ is a mapping assigning to any $\alpha : [a, b] \to M$ the parallel transport P^α along α generated by ∇.

Let D be the derivation along paths corresponding to ∇ according to Corollary 3.1. Then (3.20) holds for $\gamma = \alpha$, so (3.23) is tantamount to

$$D^\alpha_s \hat u = 0, \qquad u(a) = u_0, \tag{3.24}$$

where $\hat u : \alpha \mapsto \bar u \circ \alpha$ with $\bar u \in \mathrm{Sec}^1(E, \pi, M)$ such that $\bar u|_{\alpha([a,b])} = u$. From here and the results of [8, Section 2] immediately follows that the lifting $\hat u$ is generated by the unique linear transport P along paths corresponding to D,

$$\hat u : \alpha \mapsto \hat u_\alpha := \bar{\mathsf{P}}^\alpha_{a,u_0}, \quad \bar{\mathsf{P}}^\alpha_{a,u_0} : s \mapsto \bar{\mathsf{P}}^\alpha_{a,u_0}(s) := \mathsf{P}^\alpha_{a\to s} u_0, \quad s \in [a, b]. \tag{3.25}$$

Therefore, $P^\alpha(u_o) := u(b) = \bar u(\alpha(b)) = \hat u_\alpha(b) = \mathsf{P}^\alpha_{a\to b} u_0$. Since this is valid for all $u_0 \in \pi^{-1}(\alpha(a))$, we have

$$P^\alpha = \mathsf{P}^\alpha_{a\to b}. \tag{3.26}$$

Theorem 3.2. *The parallel transport P generated by a linear connection ∇ in a vector bundle coincides, in a sense of (3.26), with the unique linear transport P along paths in this bundle corresponding to the derivation D along paths defined, via Corollary 3.1, by the connection. Conversely, if P is a linear transport along paths whose coefficients' matrix admits the representation (3.21), then for every $s, t \in [a, b]$*

$$\mathsf{P}^\alpha_{s\to t} = \begin{cases} P^{\alpha|[s,t]} & \text{for } s \le t, \\ (P^{\alpha|[t,s]})^{-1} & \text{for } s \ge t, \end{cases} \tag{3.27}$$

[5] If γ is not injective, here and henceforth $(\nabla_{\dot\gamma}\sigma)|_{\gamma(s)}$ should be replaced by $D^\gamma_s \hat\sigma$, $\hat\sigma : \gamma \mapsto \sigma \circ \gamma$, where D is the derivation along paths corresponding to ∇ via Corollary 3.1.

where P is the parallel transport along paths generated by the unique linear connection ∇ corresponding to the derivation D along paths defined by P.

Proof. The first part of the assertion was proved above while deriving (3.26). The second part is simply the inversion of all logical links in the first one, in particular (3.27) is the solution of (3.26) with respect to P. □

Remark 3.3. The (local) condition (3.21) plays a crucial role in the proofs of all of the above results. It has also an invariant version in terms of linear transports: for a given transport L it is "almost" equivalent to the conditions $L^{\gamma\circ\varphi}_{s\to t} = L^{\gamma}_{\varphi(s)\to\varphi(t)}$, $s,t \in J''$, and $L^{\gamma|J'}_{s\to t} = L^{\gamma}_{s\to t}$, $s,t \in J'$, where $\gamma : J \to M$, $\varphi : J'' \to J$ is orientation-preserving diffeomorphism, and $J' \subseteq J$ is a subinterval. This means that, in some sense, a linear transport L is a parallel one (generated by a linear connection) iff it satisfies these conditions. These statements will be completely rigorous if the transports involved satisfy some smoothness conditions; for details, see [23]. A revised and expanded study of the links between linear and parallel transports will be given elsewhere.

The transport P along paths corresponding according to Theorem 3.2 to a parallel transport P or a linear connection ∇ will be called *parallel transport along paths*.

Corollary 3.2. *The local coefficients' matrix Γ of a parallel transport along paths and the coefficients' matrices Γ_μ of the generating it* (*or generated by it*) *linear connection are connected via* (3.21) *for every C^1 path $\gamma : J \to M$.*

Proof. See Theorem 3.2. □

3.4. Normal frames for linear connections

In Section 2, problems of existence, uniqueness, and holonomicity of frames normal for derivations of the tensor algebra over a manifold were completely solved on arbitrary submanifolds. In particular, all of these results apply for linear connections on manifolds, i.e. for linear connections in the tangent bundle over a manifold (see Section 2.5). The purpose of this section is to be obtained similar results for linear connections in arbitrary finite-dimensional vector bundles whose base and bundle spaces are C^∞ manifolds. The method we are going to follow is quite simple: relying on the conclusions of the previous sections, we shall transfer the general results of [8], concerning frames normal for linear transports, and of Section 2 to analogous ones regarding linear connections in vector bundles. More precisely, the methods of Sections 5-7 of [8] should be applied as (3.21) holds for parallel transports generated by linear connections. Equivalently well, as we shall see, the methods and results of Section 2 and of [2–4] can almost directly be used.

Definition 3.4. Given a linear connection ∇ in a vector bundle (E,π,M) and a subset $U \subseteq M$. A *frame* $\{e_i\}$ in E defined over an open subset V of M containing U or equal to it, $V \supseteq U$, is called *normal for ∇ over U* if in it and some (and hence any) frame $\{E_\mu\}$ in

$T(M)$ over V the coefficients of ∇ vanish everywhere on U. Respectively, $\{e_i\}$ is *normal for* ∇ *along a mapping* $g : Q \to M$, $Q \neq \emptyset$, if $\{e_i\}$ is normal for ∇ over $g(Q)$.

If one wants to attack directly the problems for existence, uniqueness, etc. of frames normal for a linear connection ∇, the transformation formula (3.5) should be used. Indeed, if $(\{e_i\}, \{E_\mu\})$ is an arbitrary pair of frames over $V \supseteq U$, a frame $\{e'_i = A_i^j e_j\}$ is normal for ∇ over U if for some $\{E'_\mu = B_\mu^\nu E_\nu\}$ in the pair $(\{e'_i\}, \{E'_\mu\})$ is fulfilled $\Gamma'^i_{j\mu}|_U = 0$, which, by (3.5), is equivalent to

$$(\Gamma_\nu A + E_\nu(A))|_U = 0. \tag{3.28}$$

We call this (matrix) equation the *equation of the normal frame(s)* for ∇ over U or simply the *normal frames equation* (for ∇ on U). It contains all the information for the frames normal for a given linear connection, if any. Since in (3.28) the matrix $B = [B_\mu^\nu]$ does not enter, a trivial but important corollary of it is that the choice of the frame $\{E_\mu\}$ over V in $T(M)$ is completely insignificant in a sense that if in $(\{e'_i\}, \{E'_\mu\})$ the coefficients of ∇ vanish on U, then they also have this property in $(\{e'_i\}, \{E''_\mu\})$ for any other frame $\{E''_\mu\}$ over V in $T(M)$.

If one likes, he/she could begin an independent investigation of the normal frames equation (3.28) with respect to the C^1 non-degenerate matrix-valued function A which performs the transition from an arbitrary fixed (chosen) frame $\{e_i\}$ to normal ones, if any. But we are not going to do so since this equation has been completely studied in the practically most important (at the moment) cases, the only thing needed is the existing results to be carried across to linear connections.

Recall [8, Definition 7.2], a frame $\{e_i\}$ is called strong normal on U for a linear transport L along paths, for which (3.21) holds, if in $(\{e_i\}, \{E_\mu\})$ for some frame $\{E_\mu\}$ the 3-*index coefficients' matrices* Γ_μ of L vanish on U.

Proposition 3.1. *The frames normal for a linear connection in a vector bundle are strong normal for the corresponding to it parallel transport along paths and vice versa.*

Proof. See Corollary 3.2. □

As we pointed in [8, Section 7], the most interesting problems concerning strong normal frames are practically solved in Section 2 (see also in [2–4]). Let us repeat the arguments for such a conclusion and state, due to Proposition 3.1, the main results in terms of linear connections in vector bundles.

Let (E, π, M) be finite-dimensional vector bundle with E and M being C^∞ manifolds, $U \subseteq M$, $V \subseteq M$ be an *open* subset containing U, $V \supseteq U$, and ∇ be linear connection in (E, π, M). The problem is to be investigated the frames normal for ∇ over U or, equivalently, the ones strong normal for the parallel transport along paths generated by ∇.

Above we proved that a frame $\{e'_i\}$ over V in E is normal for ∇ over U if and only if for arbitrarily fixed pair of frames $(\{e_i\}, \{E_\mu\})$, $\{e_i\}$ in E and $\{E_\mu\}$ in $T(M)$, over V there is a non-degenerate C^1 matrix-valued function A satisfying (3.28), in which Γ_μ are the coefficients' matrices of ∇ in $(\{e_i\}, \{E_\mu\})$, and such that $e'_i = A_i^j e_j$. In other words, $\{e'_i\}$ is normal for ∇ over U if it can be obtained from an arbitrary frame $\{e_i\}$ via transformation whose matrix is a solution of (3.28).

Comparing Eq. (3.28) with (2.9), we see that, in view of (2.10), they are identical with the only difference that the size of the square matrices $\Gamma_1, \dots, \Gamma_{\dim M}$, and A in Section 2 is $\dim M \times \dim M$, while in (3.28) it is $v \times v$, where v is the dimension of the vector bundle (E, π, M), i.e. $v = \dim \pi^{-1}(x)$, $x \in M$, which is generally not equal to $\dim M$. But this difference is completely insignificant from the view-point of solving these equations (in a matrix form) or with respect to the integrability conditions for them. Therefore all of the results of [2–4], concerning the solution of the matrix differential equation (3.28), are (mutatis mutandis) applicable to the investigation of the frames strong normal on a set $U \subseteq M$.

The transferring of the results from Section 2 and from [2–4] is so trivial that their explicit reformulations has a sense if one really needs the corresponding rigorous assertions for some concrete purpose. By this reason, we want to describe below briefly the general situation and one its corollary.

Theorem 3.3 (see Theorem 2.1). *If $\gamma_n : J^n \to M$, J^n being neighborhood in $\mathbb{R}^n$, $n \in \mathbb{N}$, is a C^1 injective mapping, then a necessary and sufficient condition for the existence of frame(s) normal over $\gamma_n(J^n)$ for some linear connection in a vector bundle (E, π, M) is, in some neighborhood (in $\mathbb{R}^n$) of every $s \in J^n$, their (3-index) coefficients to satisfy the equations*

$$(R_{\mu\nu}(-\Gamma_1 \circ \gamma_n, \dots, -\Gamma_{\dim M} \circ \gamma_n))(s) = 0, \qquad \mu, \nu = 1, \dots, n, \tag{3.29}$$

where $R_{\mu\nu}$ (in a coordinate frame $\{E_\mu = \partial/\partial x^\mu\}$ in a neighborhood of $x \in M$) are given via

$$R_{\mu\nu}(-\Gamma_1, \dots, -\Gamma_{\dim M}) := -\frac{\partial \Gamma_\mu}{\partial x^\nu} + \frac{\partial \Gamma_\nu}{\partial x^\mu} + \Gamma_\mu \Gamma_\nu - \Gamma_\nu \Gamma_\mu.$$

for $x^\mu = s^\mu$, $\mu, \nu = 1, \dots, n$ with $\{s^\mu\}$ being Cartesian coordinates in $\mathbb{R}^n$.

From (3.29), an immediate observation follows (cf. 6 and [4, Section 5]): strong normal frames always exist at every point ($n = 0$) or/and along every C^1 injective path ($n = 1$). Besides, these are the *only cases* when normal frames *always exist* because for them (3.29) is identically valid. On submanifolds with dimension greater than or equal to two normal frames exist only as an exception if (and only if) (3.29) holds. For $n = \dim M$, Eqs. (3.29) express the flatness of the corresponding linear connection.

If on U exists a frame $\{e_i\}$ normal for ∇, then all frames $\{e'_i = A_i^j e_j\}$ which are normal over U can easily be described: for the normal frames, the matrix $A = [A_i^j]$ must be such that $E_\mu(A)|_U = 0$ for some (every) frame $\{E_\mu\}$ over U in $T(M)$ (see (3.28) with $\Gamma_\mu|_U = 0$).

These conclusions completely agree with the ones made in [8, Section 8] concerning linear connections on a manifold M, i.e. in the tangent bundle $(T(M), \pi_T, M)$.

3.5. Inertial frames and equivalence principle in gauge theories

In [24], it was demonstrated that, when gravitational fields are concerned, the inertial frames for them are the normal ones for the linear connection describing the field and they coincide with the (inertial) frames in which the special theory of relativity is valid. The last assertion is the contents of the (strong) equivalence principle. In the present section, relying on the ideas at the end of [24, Section 5], we intend to transfer these conclusions to the area of classical gauge theories.[6]

Freely speaking, an inertial frame for a physical system is a one in which the system behaves in some aspects like a free one, i.e. such a frame 'imitates' the absence (vanishment) of some forces acting on the system. Generally inertial frames exist only locally, e.g. along injective paths, and their existence does not mean the vanishment of the field responsible for a particular force. The best known example of this kind of frames, as we pointed above, is the gravitational field. Below we rigorously generalize these ideas to all gauge fields.

The gauge fields were introduced in connection with the study of fundamental interactions between elementary particles.[7] Later it was realized [20,21,27] that, from mathematical view-point, they are equivalent to the concept of (linear) connection on (principal) vector bundle which was clearly formulated a bit earlier. The present day understanding is that[8] "a gauge field is a connection on the principal fibration in which the vector bundle of the particle fields is associated. More precisely, we identify a gauge field with the connection 1-form or with its coefficients in terms of a local basis of the cotangent bundle of the base manifold". Before proceeding on with our main topic, we briefly comment on this definition of a gauge field.

The definition of a principal bundle (fibre bundle, fibration) and the associated with it vector bundle can be found in any serious book on differential geometry or its applications—e.g. in [10, Chapter I, Section 5], [28, pp. 193–204] or [20, p. 26]—and will not be reproduced here. A main feature of a principal bundle (P, π, M, G), consisting of a bundle (P, π, M) and a Lie group G, is that the (typical) fibre of (P, π, M) is G and G acts freely on P to the right.

Recall [10,20], a connection 1-form (of a linear connection) is a 1-form with values in the Lie algebra of the group G, but, for the particular case and purposes, it can be considered a matrix-valued 1-form, as it is done in [22, p. 118] (cf. [10, Chapter III, Section 7]). Let (E, π_E, M, F) be the vector bundle with fibre F associated with (P, π, M, G) and some (left) action L of G on the manifold F.[9] According to known definitions and results, which, for instance, can be found in [18–20], a connection 1-form on (P, π, M, G) induces a linear connection (more precisely, covariant derivative operator) ∇ in the associated vector

[6] The primary role of the principle of equivalence is to ensure the transition from general to special relativity. It has quite a number of versions, known as weak and strong equivalence principles [25, pp. 72–75], any one of which has different, sometimes non-equivalent, formulations. In the present paper, only the strong(est) equivalence principle is considered. Some of its formulations can be found in [24].

[7] See, e.g., the collection of papers [26].

[8] The next citation is from [22, p. 118].

[9] See, e.g. [18] or [28] for details. In the physical applications F is a vector space and L is treated as a representation of G on F, i.e. a homomorphism $L : G \to \mathrm{GL}(F)$ from G in the group $\mathrm{GL}(F)$ of non-degenerate linear mappings $F \to F$.

bundle (E, π_E, M, F) in which the particle fields 'live' as sections.[10] The local coefficients $A^i_{j\mu}$ of ∇ in some pair of frames $(\{e_i\}, \{E_\mu\})$, $\{e_i\}$ in E and $\{E_\mu\}$ in $T(M)$, represent (locally) the connection 1-form (gauge field) and are known as *vector potentials* in the physical literature.[11] Consequently, locally a gauge field can be identified with the vector potentials which are the coefficients of the linear connection ∇ (in the associated bundle (E, π_E, M, F)) representing the gauge field.

Relying on the previous experience with gravity [24], we *define the physical concept inertial frame for a gauge field to coincide with the mathematical one normal frame for the linear connection whose local coefficients* (*vector potentials*) *represent* (*locally*) *the gauge field*. This completely agrees with the said at the beginning of the present section: according to the accepted procedure [20,30], the Lagrangian of a particle field interacting with a gauge field is obtained from the one of the same field considered as a free one by replacing the ordinary (partial) derivatives with the covariant ones corresponding to the connection ∇ representing the gauge field. Therefore, in a frame inertial on a subset $U \subseteq M$ for a gauge field the Lagrangian of a particle field interacting with the gauge field coincides with the Lagrangian for the same field considered as a free one.[12] So, we can assert that in an inertial frame the physical effects depending directly on gauge field (but not on its derivatives!) disappear. From the results obtained in the present work directly follows the existence of inertial frames for a gauge field at any fixed spacetime point or/and along injective path. On other subsets of the spacetime, inertial frames may exist only as an exception for some particular gauge fields.[13]

The analogy with gravity is quite clear and it is due to the simple fact that the gravitational as well as gauge fields are locally described via the local coefficients of linear connections, in the bundle tangent to the spacetime in the former case and in some other bundle over it in the latter one. This state of affairs can be pushed further. The above-mentioned procedure for getting the non-free Lagrangian (or field equations) for a particle field interacting with a gauge one is nothing else than the *minimal coupling* (*replacement, interaction*) *principle* applied to the particular situation. As a result the free Lagrangian (or field equation) plays the role of a Lagrangian (field equation) in an inertial frame in the sense of special relativity [24]. Call a frame $\{e_i\}$, in the bundle space of the bundle associated with the principal bundle in which particle fields live, *inertial* (in a sense of special relativity) if in it the field Lagrangian (equation) is free one. Now we can formulate the *equivalence principle* in gauge field theories (cf. [24, p. 216]). It assets the coincidence of two types of inertial frames: *the normal ones in which the vector potentials of a gauge field* (*considered as linear*

[10] The explicit construction of ∇ can be found in [29, p. 245ff].

[11] For example, see [20,21,27]. Often [22,30] a particle field ψ is represented as a vector-colon (in a given frame $\{e_i\}$) transforming under a representation $L(G)$ of the structure group G in the group $\mathrm{GL}(n, \mathbb{K})$ of non-degenerate $n \times n$, $n = \dim F$, matrices over $\mathbb{K} = \mathbb{R}, \mathbb{C}$. In this case, the matrices $A_\mu = [A^i_{j\mu}]^n_{i,j=1}$ are written as $A_\mu = A^i_\mu T_i$ where T_i are the matrices (generators) forming a basis of the Lie algebra in the representation $L(G)$ and the function A^i_μ are known as Yang–Mills fields (if the connection satisfies the Yang–Mills equation) which are also identified with the initial gauge field. Sometimes the matrices A_μ are called Yang–Mills fields too and are considered as (components of) a vector field with values in the Lie algebra of $L(G)$.

[12] Generally this does not mean that in an inertial frame disappear (all of) the physical effects of the gauge field as they, usually, depend on the curvature of describing it linear connection. Besides, it is implicitly supposed the Lagrangians to depend on the particle fields via them and their first derivatives.

[13] On submanifolds these special fields are selected by Theorem 3.3.

connection) *vanish and the inertial frames in which the Lagrangian* (*field equation*) *of a particle field interacting with the gauge one is free*.[14] According to the above discussion *the equivalence principle is a theorem*, not an axiom, in gauge theories as one can expected from a similar result in gravity.

Consequently, we have a separate equivalence principle for each gauge field. Can we speak of a single equivalence principle concerning simultaneously *all gauge fields and gravity*? The answer is expected (in a sense) to be positive. However, its argumentation and explanation depends on the particular theory one investigates since at this point we meet the problem of unifying the fundamental interactions described mathematically via linear connections in vector bundles. Below we outline the most simple situation, which can be called a 'direct sum of the interactions' and does not predict new physical phenomena but on its base one may do further research on the subject.

Suppose a particle field ψ interacts with (independent) gauge fields represented as linear connections $\nabla^{(a)}$, $a = 1, \dots, n$, $n \in \mathbb{N}$, acting in vector bundles $\xi_a := (E_a, \pi_a, M,)$, $a = 1, \dots, n$, respectively, with M being a manifold used as spacetime model. To include the gravity in the scheme, we assume it to be describe by a (possibly (pseudo-)Riemannian) linear connection $\nabla^{(0)}$ on M, i.e. in the tangent bundle $\xi_0 := (T(M), \pi_T, M) = (E_0, \pi_0, M)$. Let $\xi := (E, \pi, M \times \cdots \times M)$, where M is taken $n+1$ times, be the direct sum [18,21] of the bundles $\xi_0, \dots, \xi_n$.[15] In this case, the particle field ψ should be considered as a section $\psi \in \mathrm{Sec}^2(\xi)$ and the system of gauge fields with which it interacts is represented by a connection ∇ equal to the direct sum of $\nabla^{(0)}, \dots, \nabla^{(n)}$, $\nabla = \nabla^{(0)} \times \cdots \times \nabla^{(n)}$ (see, e.g. [21, p. 254]).

Now the minimal coupling principle says that the non-free Lagrangian of ψ is obtained from the free one by replacing in it the partial derivatives with covariant ones with respect to ∇. Since the fields with which ψ interacts are supposed independent, the frames inertial for them, if any, are completely independent. Therefore if for some set $U \subseteq M$ and any $a = 0, \dots, n$ there is a frame $\{e^{(a)}_{i_a} : i_a = 1, \dots, \dim \pi_a^{-1}(x), x \in M\}$ normal over U for $\nabla^{(a)}$, the direct product of these frames, $\{e_i := 1, \dots \dim \pi^{-1}(x), x \in M\} := \{\epsilon^{(0)}_{i_0} \times \cdots \times \epsilon^{(n)}_{i_n}\}$, is a frame normal over U for ∇. In this sense, $\{\epsilon_i\}$ is an inertial frame for the considered system of fields. We can assert the existence of such frames at any point in M and/or along any injective path in M. Now the principle of equivalence becomes the trivial assertion that inertial frames for the system of fields coincide with the normal ones for ∇.

4. Conclusion

In this paper have been found/reviewed a number of important results concerning existence, uniqueness, holonomicity, construction, etc. of frames normal for derivations of the tensor algebra over a differentiable manifold. They turn to be mutatis mutandis valid for linear connections in vector bundles. A particular example for that being Theorem 3.3

[14] Notice, here and above we do not suppose the spacetime to be flat.

[15] For purposes which will be explained elsewhere the direct sum of the mentioned bundles should be replace with the bundle $(\mathsf{E}, \boldsymbol{\pi}, M)$ where $\mathsf{E} := \{(u_0, \dots, u_n) \in E_0 \times \cdots \times E_n : \pi_0(u_0) = \cdots = \pi(u_n)\}$ and $\boldsymbol{\pi}(u_0, \dots, u_n) := \pi_0(u_0)$ for $(u_0, \dots, u_n) \in E$. So that $\pi^{-1}(x) = \pi_0^{-1}(x) \times \cdots \times \pi_n^{-1}(x)$ for $x \in M$.

from which follows that any linear connection in a vector bundle admits frames normal at a single point or/and along an injective path. As a consequence, as we saw, the concept of an inertial frame (of reference), usually associated to systems in gravitational field, can be transferred to the area of gauge theories which, in turn, allows the extension of the range of validity of the principle of equivalence for gravitational physics to systems interacting via gauge fields (and, of course, gravitationally).

We would like to say that the physical importance of the normal frames, more precisely of normal coordinates, was notice in different directions already in the early works on normal coordinates, like [1,31].

At the end, we shall mention the geometric equivalence principle (see [25, p. 76], [32, p. 19], [33, p. 3]): there are reference frames with respect to which Lorentz invariants can be defined everywhere on the spacetime and that are constant under parallel transport. A possible item for further research is to replace here the Lorentz invariants with the ones (of a representation) of the structure group of a gauge theory which will lead to the transferring of the (geometric) equivalence principle to the gauge theory whose structure group is involved.

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